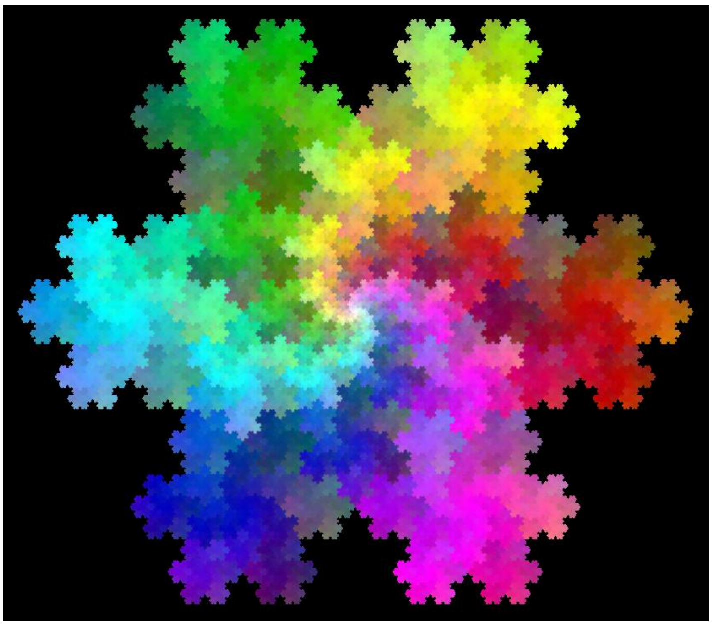
The Koch Snowflake

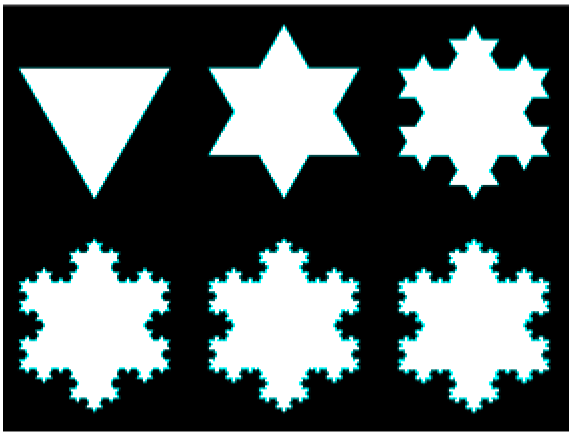


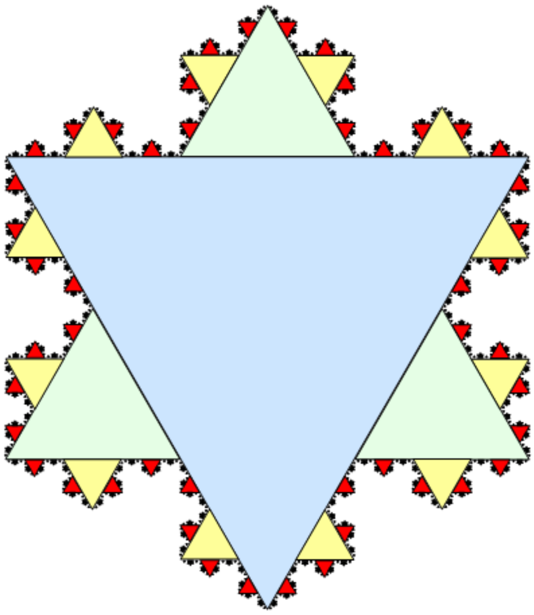
https://readersquest.files.wordpress.com/2011/06/koch-1-14.png

The Koch Snowflake is a wonderful example of how beautiful mathematics is, and leads to all sorts of interesting ideas and questions. One of the questions you might ask about any 2D shape is the value of its perimeter and area. What sounds like a simple GCSE question, is actually quite tricky when thinking about the Koch Snowflake!

You might be looking at this shape, and think how is it even possible? It has so many wiggly bits, it looks impossible to calculate anything! Like with many tricky ideas in mathematics, it can help to simplify the problem and look for patterns!

Let us look at how the Snowflake is constructed:





https://wiki.ubc.ca/File:Koch_Snowflake_Triangles.png

https://i.pinimg.com/originals/91/ba/e8/91bae8ae4dd0f0b914d077ee92cda1ca.png

From the images above, it is possible to see the first few individual steps, or iterations, of the Snowflake. You start with a triangle, and add 3 smaller ones to the middle of the sides of your starting triangle. You then add 12 even smaller triangles onto the side lengths of the 3 smaller triangles etc. etc.

So to work out the area, all we need to do is add lots of triangles together. In fact, due to the nature of the Snowflake, we are going to have to add infinitely many triangle together….oh dear, this could take a while – time to look for patterns to save us some time!

The size of the starting triangle can vary, so to keep it simple, let us say that the area of the large blue triangle is 1 (referring to the second picture above). Now try and work your way through the following questions to reach the final solution!

[NB: for most questions below there is a hint right at the end to help if you are stuck]

Question 1: If the large blue triangle has area 1, what is the total area of the three green triangles (leave your answer as a fraction)?

Question 2: what about the total area of just the yellow triangles? What about just the red ones? (again, leave your answers as fractions).

You should now have 3 values: the area of the green triangles, the area of the yellow triangles, and the area of the red ones. You could try to find the next value to add on, but perhaps this is already enough to spot a pattern.

Perhaps you would like to try to work out the pattern before reading on?

The sequence of numbers you have generated is called a geometric sequence. A geometric sequence is one where to get to the next term, you *multiply* the current term by a fixed number.

For example, 3, 6, 12, 24, 48, …… each term has been multiplied by 2, in order to get to the next term. Another example is 48, 12, 3, 0.75…. Where each term has been multiplied by $\frac{1}{4}$. The value of "2" or "$\frac{1}{4}$" is called the common ratio for the geometric sequences.

Question 3: Find the common ratio of the following geometric sequences:

a) -2, -14, -98, -686, …
b) 0.5, -2, 8, -32, …
c) 243, 81, 27, 9, …
d) 6, $\frac{12}{5}$, $\frac{24}{25}$, $\frac{48}{125}$, …

Question 4: What is the common ratio for the sequence of triangle areas that you calculated in questions 1 and 2?

Now that we have worked out the pattern, it is time to add it all up. For that, we need to know how to add up the terms of a geometric sequence – thankfully, there is a formula to help us!

The sum of the first 'n' terms of a geometric sequence is calculated using the formula:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Where $'a'$ is the first term of the sequence, $'r'$ is the common ratio, and $'n'$ is the number of terms you want to add together.

Question 5: Calculate the sum of the following geometric sequences:

a) The first 8 terms of the sequence 1, 2, 4, 8, …
b) The First 100 terms of $-\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}$, …
c) The first 5 terms of 5, 4.5, 4.05, …
d) The first 10 terms if the first term is 9, and the common ratio is $\frac{2}{3}$ (round your answer to 2dp)

Excellent, so now we have a real chance at calculating the area of the Koch Snowflake, the only trouble is though, that we need to add together infinitely many triangles, not just a specific amount. Thankfully, this has been thought of as well with its own formula.

The conceptual step, is the idea that if the value of r is between -1 and 1, then taking higher powers of r, for example r^{100} will give a really, really, really small number, more or less 0 even. Give it a try, although your calculator will likely have to provide the answer in standard form or even say "0".

So if r is between -1 and 1,and n is a massive number, r^n will be so small we can ignore it, which leads to the new formula for summing to infinity:

$$S_\infty = \frac{a}{1-r}$$

Question 6: Find the sum to infinity for the following geometric sequences:

a) 1, 0.1, 0.01, …
b) 10, -5, 2.5, -1.25, …
c) First term 5 and common ratio 0.6

Question 7: Now it is time to find the sum to infinity of the sequence we created for the area of the smaller triangles which.

Finally, remember to add "1", which was the area of the starting blue triangle, and ta da – you have the area of the whole Koch snowflake!

This task involved a whole range of skills, mainly learnt in A-level maths, so well done for completing it!

Hints, solutions and further thoughts

I leave it as a challenge to you to find the length of the perimeter of the Koch Snowflake, which could lead you to the famous coastline paradox. I will leave one last hint for in the hints section below.

Len Brin has also done a nice interactive visual for both the area and perimeter using Geogebra which you can find here: https://www.geogebra.org/m/J6PHUJWs

Hints

[hint to Q1: how many green triangles fit into the large blue one?]

[hint to Q2: to calculate the fraction of a fraction, you multiply the two fractions together, for example to find $\frac{1}{4}$ *of* $\frac{1}{3}$ *, you would do* $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$ *]*

[hint for Q3: chose any term, and divide it by the previous term]

[hint for Q5: first make a note of the values for a, n and r. Then substitute into the formula]

[final hint: the sum to infinity formula only *works if the ratio r is between -1 and 1!]*

Proof of the S_n formula

1	$S_n = a + ar + ar^n + \cdots + ar^{n-2} + ar^{n-1}$	Show that the sum of a geometric series can be found by adding up each of the individual terms.
2	$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$	Multiply S_n by r.
3	$S_n - rS_n = a - ar^n$	Subtract rS_n from S_n.
4	$S_n(1-r) = a(1-r^n)$	Factorise each side of the equation.
5	$S_n = \dfrac{a(1-r^n)}{1-r}$	Divide each side by $(1-r)$.

https://reviseblog.files.wordpress.com/2015/05/sum-of-a-geometric-series-proof.png

Solutions

Q1) 1/3 Q2) 4/27 and 16/243 Q3) a) 7 b) -2 c) 1/3 d) 2/5

Q4) 4/9 Q5) a)255 b) -5 c)20.4755 d)26.53 Q6) a)10/9 b)20/3 c) 12.5

Q7) 8/5